JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **201,** 642–659 (1996)
ARTICLE NO. 0279

Approximating Properties of Entire Functions of Exponential Type

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In this paper, we prove that

$$E\big(f, \mathscr{E}_\sigma \cap L_p(\mathbb{R})\big)_{p(\mathbb{R})} = \lim_{m\to\infty} E\big(f, \mathscr{S}_{\pi/\sigma, m} \cap L_p(\mathbb{R})\big)_{p(\mathbb{R})}, \qquad 1 < p < \infty,$$

$$E\big(f, \mathscr{E}_\sigma \cap L_p(\mathbb{R})\big)_{p(\mathbb{R})} \le \underline{\lim}_{m\to\infty} E\big(f, \mathscr{S}_{\pi/\sigma, m} \cap L_p(\mathbb{R})\big)_{p(\mathbb{R})}, \qquad p = 1, \infty,$$

$$E\big(f, \mathscr{E}_{\sigma'} \cap L_p(\mathbb{R})\big)_{p(\mathbb{R})} \ge \overline{\lim}_{m\to\infty} E\big(f, \mathscr{S}_{\pi/\sigma, m} \cap L_p(\mathbb{R})\big)_{p(\mathbb{R})},$$

$$p = 1, \infty;\ 0 < \sigma' < \sigma,$$

where $\mathscr{S}_{\pi/\sigma, m}$ denotes the space of cardinal splines of degree m with nodes $\{[\nu + \frac{1}{2}(m-1)h]\}_{j\in\mathbb{Z}}$, and $\mathscr{E}_\sigma$ denotes the restriction to the real line $\mathbb{R}$ of entire functions of exponential type σ. From this connection, we solve two extremal problems of some fundamental classes of functions defined of $\mathbb{R}$. © 1996 Academic Press, Inc.

1. INTRODUCTION

Both entire functions of exponential type and cardinal splines are fundamental approximating tools for the classes of functions defined on $\mathbb{R}$. On the one hand, Bernstein, Krein, and Akhieser [1] and many others studied the extremal properties of entire functions of exponential type and obtained many important results. On the other hand, in the seventies, Schoenberg [2], Karlin, Micchelli and Pinkus [3], Marsden, Richards, and Riemenschneider [4], and many others investigated the properties of cardinal splines, and their research gives another new and powerful tool for the approximation of classes of functions defined on $\mathbb{R}$. Recently, after

Supported by Natural Science Fundation of P. R. China.

0022-247X/96 $18.00

the concept of average width was introduced by Tikhomirov [5], some works (see [6–8] and the references of [6]) have demonstrated that both entire functions of exponential type and cardinal splines are optimal in the sense of average width for some fundamental classes of functions defined on $\mathbb{R}$. It is the purpose of the present paper to establish some connection between approximating properties of the entire functions of exponential type and cardinal splines. The interest of this result lies in the fact that we may solve some extremal problems of approximation of some classes of functions defined on $\mathbb{R}$ by entire functions of exponential type.

DEFINITION 1.1. Let m be a non-negative integer, and let $\mathscr{S}_{h,m} = \{s\}$ denote the class of functions s, satisfying the following conditions: (i) $s \in C^{m-1}(\mathbb{R})$, (ii) the restriction of s to every interval $[\nu h + \frac{1}{2}(m-1)h, (\nu + 1)h + \frac{1}{2}(m-1)h)$ is a polynomial of degree not exceeding m.

In particular, for $h = 1$, we write also $\mathscr{S}_m$ instead of $\mathscr{S}_{1,m}$. We denote by $\tilde{\mathscr{S}}_{1/n,m}$, $n \in \mathbb{N}$, the 2-periodic subspace of $\mathscr{S}_{1/n,m}$.

Let E be a finite interval or $\mathbb{R}$; we define $L_p(E)$, $1 \le p \le \infty$, as the classical Lebesgue space on E, $\|\cdot\|_{p(E)}$ denotes the norm of $L_p(E)$, and for convenience, we write also

$$\|\cdot\|_p := \|\cdot\|_{p(\mathbb{R})},$$

and finally let

$$\mathscr{S}_{h,m,p} = \left\{s \in \mathscr{S}_{h,m} : s \in L_p(\mathbb{R})\right\}, \qquad \mathscr{S}_{m,p} := \mathscr{S}_{1,m,p}. \tag{1.1}$$

Denote by $L_p^r(E)$, $1 \le p \le \infty$, the subspace of function f in $L_p(E)$ for which the $(r-1)$st derivation of f exists, is absolutely continuous on E, and $\|f^{(r)}\|_{p(E)}$ is finite; further,

$$W_p^r(E) := \left\{f \in L_p^r(E) : \|f^{(r)}\|_{p(E)} \le 1\right\}. \tag{1.2}$$

Let $\tilde{L}_p, \tilde{W}_p^r$ be the 2-periodic subsets of $L_p[-1,1]$ and $W_p^r[-1,1]$, respectively, and let $l_p, 1 \le p \le \infty$, be the Banach space of double infinite bounded sequences with the usual norm

$$\begin{aligned} \|\{y_j\}\|_p &:= \left(\sum_{j\in\mathbb{Z}} |y_j|^p\right)^{1/p}, \qquad 1 \le p < \infty, \\ \|\{y_j\}\|_\infty &= \sup_{j\in\mathbb{Z}} |y_j|. \end{aligned} \tag{1.3}$$

Further, $\mathscr{E}_\sigma(\mathbb{R})$, $\sigma > 0$, denotes the restriction on $\mathbb{R}$ of entire functions of exponential type σ, and let

$$B_{\sigma,p} = \mathscr{E}_\sigma(\mathbb{R}) \cap L_p(\mathbb{R}), \qquad 1 \le p \le \infty, \qquad B_\sigma := B_{\sigma,\infty}.$$

THEOREM A [2]. (1) *For a given sequence* $y = \{y_j\}_{j\in\mathbb{Z}} \in l_p$, $1 \le p \le \infty$, *there is a unique element* $\mathscr{L}_m y \in \mathscr{S}_{m,p}$, *interpolating the given data at the integers, i.e.*,

$$\mathscr{L}_m y(j) = y_j, \qquad j \in \mathbb{Z},$$

and

$$\mathscr{L}_m y(x) = \sum y_j L_m(x-y), \tag{1.4}$$

where $L_m(x) \in \mathscr{S}_{m,p}$ *is the fundamental spline of* $\mathscr{S}_m$ *which interpolates the data* $y_0 = 1$, $y_j = 0$, $j \ne 0$.

(2) *If* $s \in \mathscr{S}_{m,p}$, $1 \le p \le \infty$, *then* $\{s_m(j)\}_{j\in\mathbb{Z}} \in l_p$.

In [4], the behavior of $\|\mathscr{L}_m y\|_p$, $y = \{y_j\}_{j\in\mathbb{Z}} \in l_p$, p-fixed, as m tends to infinity, is investigated, and the following theorem, among others, is proved.

THEOREM B [4]. *Let* $y = \{y_j\}_{j\in\mathbb{Z}} \in l_p$, $1 < p < \infty$. *Then*

$$\lim_{m\to\infty} \left\| \sum_{j\in\mathbb{Z}} y_j L_m(x-j) - \sum_{j\in\mathbb{Z}} y_j \frac{\sin \pi(x-j)}{\pi(x-j)} \right\|_p = 0, \tag{1.5}$$

where $\sum_{j\in\mathbb{Z}} y_j((\sin \pi(x-j))/\pi(x-j))$ *is usually called the Whittaker cardinal series.*

DEFINITION 1.2. Let X be a normed linear space, for a given f and a subset $\mathfrak{M}$ of X; the quantities

$$E(f,\mathfrak{N})_X = \inf_{g\in\mathfrak{N}} \|f-g\|_X, \qquad E(\mathfrak{M},\mathfrak{N})_X = \sup_{f\in\mathfrak{M}} E(f,\mathfrak{N})_X$$

are called the best approximations of the element f and the subset $\mathfrak{M}$ by subspace $\mathfrak{N}$ of X, respectively.

It is well known that the subspace of trigonometric polynomials and periodic splines are both important approximation tools for the classes of periodic functions. Velikin [9] studied the connection between approximation properties of trigonometric polynomials and periodic splines and based on this relation, he solved some extremal problems of some convolution classes of periodic functions. Among others, he proved

THEOREM C [9]. *Let* $f \in \tilde{L}_p$, $1 \le p \le \infty$, $m, n \in \mathbb{N}$. *Then*

$$\lim_{m \to \infty} E\left(f, \tilde{\mathscr{S}}_{n^{-1}, m}\right)_{p[-1,1]} = E(f, T_{n-1})_{p[-1,1]}, \tag{1.6}$$

where

$$T_{n-1} = \operatorname{span}\{1, \cos \pi x, \sin \pi x, \dots, \cos(n-1)\pi x, \sin(n-1)\pi x\}.$$

From Theorem C, it is a natural conjecture that

$$\lim_{m \to \infty} E\left(f, \mathscr{S}_{\pi/\sigma, m, p}\right)_p = E\left(f, B_{\sigma, p}\right)_p, \qquad 1 \le p \le \infty, \tag{1.7}$$

but we showed in [10] that (1.7) is not valid in the case of $p = \infty$; i.e., there is a function $f \in C(\mathbb{R}) \cap L_\infty(\mathbb{R})$, such that

$$\lim_{m \to \infty} E\left(f, \mathscr{S}_{\pi/\sigma, m, \infty}\right)_\infty > E(f, B_\sigma)_\infty,$$

and we proved also that (1.7) is true in the case of $p = 2$.

2. THE MAIN RESULTS

It is the main purpose of the present paper to further investigate the asymptotic connection between the approximating properties of cardinal splines and entire functions of exponential type.

THEOREM 1. *Let* $f \in L_p(\mathbb{R})$, $m \in \mathbb{N}$, $\sigma > 0$. *Then*

(1) $E(f, B_{\sigma,p})_p = \lim_{m \to \infty} E(f, \mathscr{S}_{\pi/\sigma, m, p})_p$, $1 < p < \infty$;

(2) $E(f, B_{\sigma,p})_p \le \underline{\lim}_{m \to \infty} E(f, \mathscr{S}_{\pi/\sigma, m, p})_p$, $p = 1, \infty$;

(3) $E(f, B_{\sigma',p})_p \ge \overline{\lim}_{m \to \infty} E(f, \mathscr{S}_{\pi/\sigma, m, p})_p$, $0 < \sigma' < \sigma$, $p = 1, \infty$.

THEOREM 2. *If* $\mathfrak{M}$ *is a bounded subset of* $L_p(\mathbb{R})$, $1 \le p \le \infty$, *then*

$$E\left(\mathfrak{M}, B_{\sigma,p}\right)_p \le \underline{\lim}_{m \to \infty} E\left(\mathfrak{M}, \mathscr{S}_{\pi/\sigma, m, p}\right)_p.$$

Remark 2.1. It is worth pointing out that the proof of Theorem C used some characteristic properties of finite dimensional spaces, such as the compactness of the unit ball of finite dimensional space. Since $\mathscr{S}_{h,m,p}$ and $B_{\sigma,p}$ are both infinite dimensional, the proof of Theorem 1 needs new ideas and new methods, and we want to mention also that the method of proof of Theorem 1 is different from that of [10], where we used some particular properties of Hilbert spaces, such as Plancherel's theorem and Riesz-Fischer's theorem.

Some extremal problems of some classes of functions defined on $\mathbb{R}$ could be determined by Theorems 1 and 2. Let us consider some examples.

We denote by $X(\mathbb{R})$ a normed linear space defined on $\mathbb{R}$. For $f \in X(\mathbb{R})$, let $f_N(x) = f(x) \cdot \chi_N(x)$, and

$$X(I_N) := \{f_N : f \in X(\mathbb{R})\},$$

where $\chi_N(x)$, $N > 0$, is the characteristic function of the interval $[-N, N]$, and A is a linear subspace of $X(\mathbb{R})$. Set

$$K(\varepsilon, N, A) = \min\Big\{\dim F : F \subset X(I_N), E\big(B(A) \cap X(I_N), F\big)_{X(I_N)} < \varepsilon\Big\}, \quad \varepsilon > 0,$$

where $B(A)$ denotes the unit ball of A in $X(\mathbb{R})$, and $\dim F$ denotes the dimension of the finite dimensional subspace F.

Definition 2.1 [5]. Let $\sigma > 0$, and let A be a linear subspace of X. If

$$\lim_{\varepsilon \to 0} \overline{\lim_{N \to \infty}} \frac{K(\varepsilon, N, A)}{2N} = \sigma < +\infty,$$

then σ is said to be the average dimension of A in $X(\mathbb{R})$, and is denoted by $\overline{\dim} A = \sigma$.

Definition 2.2 [6]. Given a subset $\mathfrak{M} \subset X(\mathbb{R})$, $\sigma > 0$, the quantity

$$\bar{d}_\sigma(\mathfrak{M}, X(\mathbb{R})) = \inf_{\overline{\dim} A \le \sigma} E(\mathfrak{M}, A)_{X(\mathbb{R})}$$

is said to be the average σ-width in the sense of Kolmogorov. If there is a subspace A^* with average dimension $\le \sigma$ such that

$$\bar{d}_\sigma(\mathfrak{M}, X(\mathbb{R})) = E(\mathfrak{M}, A^*)_{X(\mathbb{R})},$$

then A^* is said to be an optimal subspace for $\bar{d}_\sigma(\mathfrak{M}, X(\mathbb{R}))$.

Theorem D [6]. *Let $r, m \in \mathbb{N}$, $m \ge r$, $p = 1, \infty$, $\sigma > 0$. Then*

$$\bar{d}_\sigma\big(W_p^r(\mathbb{R}), L_p(\mathbb{R})\big) = E\big(W_p^r(\mathbb{R}), \mathscr{S}_{1/\sigma, m, p}\big)_p = \frac{\mathscr{K}_r}{(\sigma\pi)^r},$$

and $\mathscr{S}_{1/\sigma, m, p}$ is optimal for $\bar{d}_\sigma(W_p^r(\mathbb{R}), L_p(\mathbb{R}))$, where $\mathscr{K}_r$ is Favard's constant, i.e.,

$$\mathscr{K}_r = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}}. \tag{2.1}$$

THEOREM E [11]. *Let $\sigma > 0$, and $1 \le p \le \infty$. Then*

$$\overline{\dim}\big(B_{\sigma,p}, L_p(\mathbb{R})\big) = \frac{\sigma}{\pi}.$$

THEOREM 3. *Let $p = 1, \infty$, $r \in \mathbb{N}$, $\sigma > 0$. Then*

$$\bar{d}_\sigma\big(W_p^r(\mathbb{R}), L_p(\mathbb{R})\big) = E\big(W_p^r(\mathbb{R}), B_{\sigma\pi,p}\big)_p = \frac{\mathcal{K}_r}{(\sigma\pi)^r}. \tag{2.2}$$

Proof. By Theorems 1 and D, we have

$$E\big(W_p^r(\mathbb{R}), B_{\sigma\pi,p}\big)_p \le \frac{\mathcal{K}_r}{(\sigma\pi)^r},$$

and from Theorems D and E, we obtain

$$E\big(W_p^r(\mathbb{R}), B_{\sigma\pi,p}\big)_p \ge \bar{d}_\sigma\big(W_p^r(\mathbb{R}), L_p(\mathbb{R})\big) = \frac{\mathcal{K}_r}{(\sigma\pi)^r}.$$

Theorem 3 is proved. ∎

The exact estimate of $E(W_p^r(\mathbb{R}), B_{\sigma,p})_p$, $p = \infty$, was determined in [1, p. 194] in a different way.

Now we consider a new extremal problem. Let f be a local p-integrable function. The Sobolev–Wiener class of functions is defined as [7, 12]

$$W_{pq}^r(\mathbb{R}) = \big\{f \in L_q^r(\mathbb{R}) : \|f^{(r)}\|_{pq} \le 1\big\},$$

where

$$\|f\|_{pq} = \Bigg\{\sum_{j\in\mathbb{Z}} \|f(\cdot + j)\|_{p[-1,1]}^q\Bigg\}^{1/q}, \qquad 1 \le p \le \infty, 1 \le q < \infty,$$

$$\|f\|_{p,\infty} = \sup_{j\in\mathbb{Z}} \|f(\cdot + j)\|_{p[-1,1]}, \qquad 1 \le p \le \infty.$$

It is clear that $W_{pp}^r(\mathbb{R}) = W_p^r(\mathbb{R})$, $1 \le p \le \infty$.

THEOREM F [7]. *Let $1 < p \le \infty$, $m, n \in \mathbb{N}$, $1/p + 1/p' = 1$. Then*

$$\bar{d}_\sigma\big(W_{p,1}^r, L_1(\mathbb{R})\big) = E\big(W_p^r(\mathbb{R}), \mathcal{S}_{n^{-1},m,1}\big)_1 = \|\Phi_{r,n}\|_{p'[0,1]},$$

where

$$\Phi_{r,n}(x) = \frac{4}{\pi(n\pi)^r} \sum_{k=0}^{\infty} \frac{\cos((2k+1)n\pi x - \pi/(2r+2))}{(2k+1)^{r+1}}$$

is the Euler spline with period $2/n$.

Using Theorems 1, E, and F, we have

THEOREM 4. *Let* $1 < p \le \infty$, $r, n \in \mathbb{N}$, $1/p + 1/p' = 1$. *Then*

$$\bar{d}_\sigma\left(W^r_{p,1}(\mathbb{R}), L(\mathbb{R})\right) = E\left(W^r_{p,1}(\mathbb{R}), B_{n\pi,1}\right)_1 = \|\Phi_{r,n}\|_{p'[0,1]},$$

and $B_{n\pi,1}$ *is optimal for* $\bar{d}_\sigma(W^r_{p,1}(\mathbb{R}), L(\mathbb{R}))$.

Remark 2.2. Theorem 4 is an analogue of Taikov's inequality on Sobolev classes $\tilde{W}^r_p, 1 < p \le \infty$, of periodic functions [13, p. 172].

We prove Theorems 1 and 2 in the case of $\sigma = \pi$ only. The general case of $\sigma > 0$ can be proved in a similar manner. In order to prove Theorem 1, we need to prove an analogue of the Markov–Bernstein inequality for cardinal splines.

3. THE MARKOV–BERNSTEIN INEQUALITY FOR CARDINAL SPLINES

LEMMA 3.1 [14]. *Let* $s \in \mathscr{S}_{m,\infty}$, $m \in \mathbb{N}$, $k = 1, \ldots, m$. *Then*

$$\|s^{(k)}\|_\infty \le \mathscr{K}_{m-k}\mathscr{K}_m^{-1}\pi^k\|s\|_\infty, \tag{3.1}$$

where $\mathscr{K}_m$ *is the Favard constant defined by* (2.1). *The constant on the right hand side of* (3.1) *cannot be improved, and the Euler spline is an extremal function of* (3.1).

LEMMA 3.2 [13, p. 125] (Stein's inequality). *Let* $f(x) \in L^m(\mathbb{R})$, $m \in \mathbb{N}$, $1 \le p < \infty$. *Then*

$$\|f^{(k)}\|_p \le C_{m,n}\|f\|_p^{1-k/m}\|f^{(m)}\|_p^{k/m}, \qquad k = 1, 2, \ldots, m-1, \tag{3.2}$$

where $C_{m,k} = \mathscr{K}_{m-k}\mathscr{K}_m^{-1+k/m}$, $\mathscr{K}_m$ *is the Favard constant.*

LEMMA 3.3. *Let* $s \in \mathscr{S}_{m,p}$, $1 \le p < \infty$, $k = 1, \ldots, m$. *Then*

$$\|s^{(k)}\|_p \le \mathscr{K}_{m-k} \cdot \mathscr{K}_m^{-1}\left(\pi\sqrt[p]{p+1}\right)^k \|s\|_p \tag{3.3}$$

$$\le 2(2\pi)^k\|s\|_p. \tag{3.4}$$

Proof. Let $s \in \mathscr{S}_{m,p}$ and $s^{(m)}(x) = c_j$, $j < x < j+1$. Then

$$\|s^{(m)}\|_p = \left(\sum_{j \in \mathbb{Z}} \int_j^{j+1} |s^{(m)}(x)|^p \, dx \right)^{1/p} = \left(\sum_{j \in \mathbb{Z}} |c_j|^p \right)^{1/p}.$$

It is not difficult to prove that, for all $j \in \mathbb{R}$,

$$\begin{aligned} \int_j^{j+1} |s^{(m-1)}(x)|^p \, dz &\geq \min_{b \in \mathbb{R}} \int_j^{j+1} |c_j x - b|^p \, dx \\ &= |c_j|^p \min_{b \in \mathbb{R}} \int_0^1 |x-b|^p \, dx \geq |c_j|^p 2^{-p} \frac{1}{p+1}; \end{aligned}$$

therefore, we have

$$\|s^{(m-1)}\|_p \geq \frac{1}{2} \sqrt[p]{\frac{1}{p+1}} \left(\sum_{j \in \mathbb{Z}} |c_j|^p \right)^{1/p} = \frac{1}{2} \sqrt[p]{\frac{1}{p+1}} \|s^{(m)}\|_p. \quad (3.5)$$

By virtue of Stein's inequality, we get

$$\begin{aligned} \|s^{(m-1)}\|_p &\leq C_{m-1,p} \|s\|_p^{1/m} \|s^{(m)}\|_p^{1-1/m} \\ &= \mathscr{K}_1 \mathscr{K}_m^{-1/m} \|s\|_p^{1/m} \|s^{(m)}\|_p^{1-1/m}. \end{aligned} \quad (3.6)$$

Since $\mathscr{K}_1 = \pi/2$ [13, p. 103], from (3.5) and (3.6), we obtain

$$\begin{aligned} \|s\|_p &\geq \mathscr{K}_m \left(\mathscr{K}_1 \|s^{(m-1)}\|_p \cdot \|s^{(m)}\|_p^{-1+1/m} \right)^m \\ &\geq \mathscr{K}_m \left(\frac{2}{\pi} \cdot \frac{1}{2} \sqrt[p]{\frac{1}{p+1}} \right)^m \|s^{(m)}\|_p; \end{aligned}$$

therefore, we have

$$\|s^{(m)}\|_p \leq \mathscr{K}_m^{-1} \left(\pi \sqrt[p]{p+1} \right)^m \|s\|_p. \quad (3.7)$$

By (3.7) and by Stein's inequality again, we have

$$\begin{aligned} \|s^{(k)}\|_p &\leq \mathscr{K}_{m-k} \mathscr{K}_m^{-1+k/m} \|s\|_p^{1-k/m} \|s^{(m)}\|_p^{k/m} \\ &\leq \mathscr{K}_{m-k} \mathscr{K}_m^{-1} \left(\pi \sqrt[p]{p+1} \right)^k \|s\|_p. \end{aligned} \quad (3.8)$$

Hence (3.3) is proved. Since [13, p. 103]

$$1 = \mathscr{K}_0 < \mathscr{K}_2 < \cdots < \mathscr{K}_4 < \cdots < \mathscr{K}_3 < \mathscr{K}_1 = \frac{\pi}{2}, \tag{3.9}$$

$$\mathscr{K}_2 = \frac{\pi^2}{8}, \qquad \mathscr{K}_3 = \frac{\pi^3}{24}, \tag{3.10}$$

$$\mathscr{K}_{m-k}\mathscr{K}_m^{-1} \leq \frac{\pi}{2} < 2, \tag{3.11}$$

which together with (3.8) complete the proof of (3.4). ∎

LEMMA 3.4 [10]. *Let* $f \in L_p^m(\mathbb{R})$, $1 \leq p < \infty$. *Then*

$$\|\{f(j)\}\|_p \leq \|f\|_p + \|f'\|_p. \tag{3.12}$$

A well known theorem of Marcinkiewicz [15] establishes the equivalence of the L_p-norm of trigonometric polynomials of order $\leq m$ and the discrete l_p^{2m-1}-norm constituted from their values at a uniform lattice [15]. We prove an analogue of this theorem for cardinal splines which will be used for the proof of Theorem 1.

THEOREM 5. *Let* $s \in \mathscr{S}_{m,p}$, $m \in \mathbb{N}$, $1 \leq p < \infty$. *Then*

$$\|\{s(j)\}\|_p \leq (1 + 4\pi)\|s\|_p.$$

Proof. From Lemmas 3.3, and 3.4, we have

$$\begin{aligned}\|\{s(j)\}\|_p &\leq \|s\|_p + \|s'\|_p \leq \|s\|_p + 4\pi\|s\|_p \\ &= (1 + 4\pi)\|s\|_p. \end{aligned} \tag{3.13}$$

Theorem 5 is proved. ∎

Remark 3.1. Schoenberg [16] proved that there is a constant $M_{m,p}$ which depends on m and p only, such that

$$\|\{s(j)\}\|_p \leq M_{m,p}\|s\|_p, \qquad 1 \leq p \leq \infty.$$

Denote by $Q_1(x)$ the characteristic function of the interval $[0,1]$, and $Q_m(x)$ the m-fold convolution of $Q_1(x)$ with itself. Explicitly, we find that

$$Q_m(x) = \frac{1}{(m-1)!}\sum_{r=0}^{m}(-1)^r\binom{m}{r}(x-r)_+^{m-1},$$

where $x_+ = \max(0, x)$, and $Q_m(x)$ has compact support $[0, m]$ and $Q_m(x) > 0$ in $(0, m)$. Finally, let $\Delta^n f(x)$ be the mth divided difference of the $f(x)$, namely,

$$\Delta^m f(x) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+k).$$

It is well known that

$$\Delta^m f(j) = \int_{\mathbb{R}} Q_m(x-j) f^{(m)}(x)\, dx. \tag{3.14}$$

Let ${}_m M(x) = Q_m(x + \frac{1}{2}m)$. Then $M_m \in \mathscr{S}_{m-1,\infty}$ and M_m has compact support $[-m/2, m/2]$, and let

$$F(z) = \sum_{\nu \in \mathbb{Z}} M_{m+1}(\nu) z^\nu, \qquad z \in \mathbb{C}$$

$$\Phi_{m+1}(u) = F(e^{iu}), \qquad i \in \sqrt{-1}, u \in \mathbb{R}. \tag{3.15}$$

Then by [16], $\Phi_{m+1}(u)$ is a cosine polynomial such that

$$\Phi_{m+1}(u) > 0 \qquad \text{for all } u \in \mathbb{R} \text{ and } \Phi_{m+1}(0) = 1, \tag{3.16}$$

$$\begin{aligned} \min_{u \in \mathbb{R}} \Phi_{m+1}(u) &= \Phi_{m+1}(\pi) \\ &= 2\left(\frac{2}{\pi}\right)^{m+1} \sum_{k=0}^{\infty} \frac{(-1)^{k(m+1)}}{(2k+1)^{m+1}} = 2^m \pi^{-m} \mathscr{K}_m, \end{aligned} \tag{3.17}$$

where $\mathscr{K}_m$ is the Favard constant defined by (2.1). It follows from (3.15) that the reciprocal of $F(z)$ admits a Laurent expansion,

$$\frac{1}{F(z)} = \sum_{-\infty}^{+\infty} \omega_\nu z^\nu, \qquad |z| = e^{iu}, \tag{3.18}$$

which is identical with the Fourier series

$$\frac{1}{\Phi_{m+1}(u)} = \sum_{\nu \in \mathbb{Z}} \omega_\nu e^{i\nu u}, \qquad \text{where } \omega_\nu = \omega_{-\nu},$$

and for all $\nu \in \mathbb{Z}, (-1)^\nu \omega_\nu > 0$; therefore,

$$\sum_{\nu \in \mathbb{Z}} |\omega_\nu| = \sum_{\nu \in \mathbb{Z}} (-1)^\nu \omega_\nu = \frac{1}{\Phi_{m+1}(\pi)} = 2^m \pi^{-m} \mathscr{K}_m. \tag{3.19}$$

For these results, we refer to [16].

Remark 3.2. Theorem 6, below, is an analogue of the Markov–Bernstein inequality which plays an important role in the proof of Theorem 1, and has also its own independent role in approximation theory.

THEOREM 6. *Let* $s \in \mathscr{S}_{m,p}$, $m \in \mathbb{N}$, $1 \le p < \infty$, $k = 1, \ldots, m$. *Then*

$$\|s^{(k)}\|_p \le (1 + 4\pi)^{k/m} \mathscr{K}_{m-k} \mathscr{K}_m^{-1} \pi^k \|s\|_p \tag{3.20}$$

$$\le 2(1 + 4\pi)\pi^k \|s\|_p. \tag{3.21}$$

Proof. We use the method of discretization of [14]. Assume $s \in \mathscr{S}_{m,p}$. Then by Lemma 3.3, $\|s^{(m)}\|_p < +\infty$, hence $\|\{s^{(m)}(j)\}\|_p < +\infty$; therefore, $s^{(m)}(x)$ is a bounded step function with discontinuities at the points $j + (m-1)/2$, for integer j. Let

$$s^{(m)}(x) = c_j, \qquad x \in [\, j + (m-1)/2, j + 1 + (m-1)/2). \tag{3.22}$$

We may therefore write

$$s^{(m)}(x) = \sum_{j \in \mathbb{Z}} c_j Q_1\left(x - j - \frac{1}{2}(m-1)\right). \tag{3.23}$$

According to [16], for $m, n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{R}$,

$$\int_{\mathbb{R}} Q_m(x - \alpha) Q_m(x - \beta)\, dx = Q_{m+n}(\alpha - \beta + m). \tag{3.24}$$

From (3.24), we have

$$\begin{aligned}\int_{\mathbb{R}} Q_m(x - i) Q_1\left(x - j - \frac{1}{2}(m-1)\right) dx \\ = Q_{m+1}\left(i - j + \frac{1}{2}(m+1)\right) = M_{m+1}(i - j).\end{aligned} \tag{3.25}$$

Using (3.14), (3.23), and (3.25), we may write

$$\begin{aligned}\Delta^m s(i) &= \int_{\mathbb{R}} Q_m(x - i) s^{(m)}(x)\, dx \\ &= \sum_{j \in \mathbb{Z}} c_j \int_{\mathbb{R}} Q_m(x - i) Q_1\left(x - j - \frac{1}{2}(m-1)\right) \\ &= \sum_{j \in \mathbb{Z}} c_j M_{m+1}(i - j).\end{aligned} \tag{3.26}$$

Since $\{c_j\} \in l_p$, we may regard the sequence convolution (3.26) as a bounded linear transformation of the space l_p into itself. Therefore, it follows from (3.16), (3.18) that the transformation (3.26) admits an inverse given by

$$c_j = \sum_{j \in \mathbb{Z}} \omega_{i-j} \, \Delta^m s(j), \qquad \text{for all } j \tag{3.27}$$

(see [14, 16]), which is the only bounded solution of (3.23).

Using (3.23), (3.27), Minkowski's inequality, and (3.27), we have

$$\begin{aligned}
\|s^{(m)}\|_p &= \left(\sum_{j \in \mathbb{Z}} |c_j|^p \right)^{1/p} \\
&= \left(\sum_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} \omega_k \, \Delta^m s(j-k) \right|^p \right)^{1/p} \\
&\leq \sum_{k \in \mathbb{Z}} \left(\sum_{j \in \mathbb{Z}} |\omega_k \, \Delta^m s(j-k)|^p \right)^{1/p} \\
&= \sum_{k \in \mathbb{Z}} |\omega_k| \cdot \left(\sum_{j \in \mathbb{Z}} \left| \sum_{r=0}^{m} (-1)^{m-r} \binom{m}{r} s(j-k+r) \right|^p \right)^{1/p} \\
&\leq \sum_{k \in \mathbb{Z}} |\omega_k| \sum_{r=0}^{m} \left(\sum_{j \in \mathbb{Z}} \left| \binom{m}{r} s(j-k+r) \right|^p \right)^{1/p} \\
&= \sum_{k \in \mathbb{Z}} |\omega_k| \sum_{r=0}^{m} \binom{m}{r} \left(\sum_{j \in \mathbb{Z}} |s(j-k+r)|^p \right)^{1/p} \\
&= 2^m \cdot 2^{-m} \pi^m \mathscr{K}_m^{-1} \cdot \|\{s(j)\}\|_p \\
&= \pi^m \cdot \mathscr{K}_m^{-1} \|\{s(j)\}\|_p .
\end{aligned} \tag{3.28}$$

Using (3.28) and Theorem 5, we obtain

$$\|s^{(m)}\|_p \leq (1 + 4\pi) \pi^m \mathscr{K}_m^{-1} \|s\|_p , \tag{3.29}$$

which together with Stein's inequality for $k = 1, 2, \dots, m-1$, gives

$$\begin{aligned}
\|s^{(k)}\|_p &\leq C_{m,k} \|s\|_p^{1-k/m} \|s^{(m)}\|_p^{k/m} \\
&\leq \mathscr{K}_{m-k} \cdot \mathscr{K}_m^{-1+k/m} (1 + 4\pi)^{k/m} \pi^k \mathscr{K}_m^{-k/m} \|s\|_p \\
&= (1 + 4\pi)^{k/m} \mathscr{K}_{m-k} \mathscr{K}_m^{-1} \pi^k \|s\|_p .
\end{aligned} \tag{3.30}$$

This completes the proof of (3.20). Now (3.30) and (3.9) show that (3.21) indeed holds. ∎

4. THE PROOFS OF THEOREMS 1 AND 2

LEMMA 4.1 [1, p. 191]. *Let* $f \in L_p^m(\mathbb{R})$, $m \in \mathbb{N}$, $1 \le p \le \infty$, $\sigma > 0$. *Then*

$$E(f, B_\sigma)_p \le \frac{\mathscr{K}_m}{\sigma^m}\|f^{(m)}\|_p \le \frac{2}{\sigma^m}\|f^{(m)}\|_p,$$

where $\mathscr{K}_m$ *is the Favard constant defined by* (2.1).

THEOREM 7. *If* $s_m \in \mathscr{S}_{m,p}$, $m = 1, 2, \ldots,$ $1 \le p \le \infty$, *and* $\|s_m\|_p \le c_0$, *then*

$$\lim_{m\to\infty} E\big(s_m, B_{\pi+\delta,p}\big)_p = 0,$$

where c_0 *is independent of* m.

Proof. From Lemma 4.1, Lemma 3.1, Theorem 6, and $\|s_m\|_p \le c_0$, we have

$$\begin{aligned} E\big(s_m, B_{\pi+\delta,p}\big)_p &\le \frac{2}{(\pi+\delta)^m}\|s_m^{(m)}\|_p \\ &\le 4(1+4\pi)\left(\frac{\pi}{\pi+\delta}\right)^m\|s_m\|_p \le 4(1+4\pi)\left(\frac{\pi}{\pi+\delta}\right)^m c_0, \end{aligned} \tag{4.1}$$

which gives

$$\lim_{m\to\infty} E\big(s_m, B_{\pi+\delta,p}\big)_p = 0.$$

Theorem 7 is proved. ∎

LEMMA 4.2. *Let* $f \in L_p(\mathbb{R})$, $1 \le p < \infty$, $\sigma > 0$, $\delta > 0$. *Then*

$$\lim_{\delta\to 0} E\big(f, B_{\sigma+\delta,p}\big)_p = E\big(f, B_{\sigma,p}\big)_p.$$

Proof. Without loss of generality, we assume $0 < \delta < \sigma$. Let g_σ and $g_{\sigma+\delta}$ be a best approximation of f by $B_{\sigma,p}$ and $B_{\sigma+\delta,p}$ in $L_p(\mathbb{R})$, respectively. It is well known that $B_{\sigma,p} \subset B_{\sigma+\delta,p}$; therefore, $g_\sigma \in B_{\sigma+\delta,p}$. From this, we have

$$\|g_{\sigma+\delta}\|_p \le \|f\|_p + \|f - g_{\sigma+\delta}\|_p \le \|f\|_p + \|f - g_\sigma\|_p \le 3\|f\|_p. \tag{4.2}$$

According to [17, p. 62] and (4.2)

$$|g_{\sigma+\delta}(x)| \le 2(\sigma+\delta)^{1/p}\|g_{\sigma+\delta}\|_p \le 6(2\sigma)^{1/p}\|f\|_p; \tag{4.3}$$

therefore, the family $\{g_{\sigma+\delta}(x)\}_{\delta>0}$ is uniformly bounded on $\mathbb{R}$ and hence $\{g_{\sigma+\delta}\}_{\delta>0}$ is a normal family. Hence, there is an analytic function $g_*(x) \in B_\sigma$, such that

$$\lim_{\delta\to 0} g_{\sigma+\delta}(x) = g_*(x) \tag{4.4}$$

uniformly on every finite interval on $\mathbb{R}$. Since $g_{\sigma+\delta}(x) \in B_{\sigma+\delta,p}$, by the results of [17, pp. 62, 194], and (4.2), we have

$$|g_{\sigma+\delta}(x)| \le 6(\sigma+\delta)^{1/p} e^{(\sigma+\delta)|y|}\|f\|_p, \qquad x, y \in \mathbb{R}, i = \sqrt{-1}, \tag{4.5}$$

which together with (4.4) gives

$$|g_*(x)| \le 6e^{\sigma|y|}\|f\|_p,$$

therefore, $g_* \in B_\sigma$. Using Fatou's lemma, and (4.2), (4.4), we have $\|g_*\|_p \le 3\|f\|_p$, hence, $g_* \in B_{\sigma,p}$. By Fatou's lemma again, we obtain

$$\begin{aligned} E(f, B_{\sigma,p})_p &\le \|f - g_*\|_p \le \varliminf_{\delta\to 0} \|f - g_{\sigma+\delta}\|_p \\ &\le \varlimsup_{\delta\to 0} \|f - g_{\sigma+\delta}\|_p \le \|f - g_\sigma\|_p = E(f, B_{\sigma,p})_p. \end{aligned}$$

This completes the proof of Lemma 4.2. ∎

Remark 4.1. In the case of $p = \infty$, an analogue of Lemma 4.2 was proved in [17, p. 58].

LEMMA 4.3 [17, p. 232] (Bernstein's inequality). *Let* $f \in B_{\sigma,p}$, $1 \le p \le \infty$, $\sigma > 0$. *Then*

$$\|f'\|_p \le \sigma\|f\|_p.$$

LEMMA 4.4 [6]. *Let* $f \in L_p^m(\mathbb{R})$, $m \in \mathbb{N}$, $p = 1, \infty$. *Then there is a unique* $\sigma_m(f, x) \in \mathscr{S}_{m,p}$ *such that* $\sigma_m(f, j) = f(j)$, $j \in \mathbb{Z}$, *and*

$$\|f(\cdot) - \sigma_m(f, \cdot)\|_p \le \frac{\mathscr{K}_m}{\pi^m}\|f^{(m)}\|_p, \tag{4.6}$$

where $\mathscr{K}_m$ *is the Favard constant.*

LEMMA 4.5. *Let* $f \in B_{\sigma,p}$, $p = 1, \infty$, $\sigma < \pi$. *Then*

$$\lim_{m\to\infty} \|f(\cdot) - \sigma_m(f,\cdot)\|_p = 0. \tag{4.7}$$

Proof. Lemma 4.5 for the case of $p = \infty$ was proved in [3, p. 294]. In a similar way and using Lemmas 4.3 and 4.4, we obtain

$$\|f(\cdot) - \sigma_m(f,\cdot)\|_p \le \frac{2}{\pi^m}\|f^{(m)}\|_p \le 2\left(\frac{\sigma}{\pi}\right)^m \|f\|_p, \tag{4.8}$$

which gives (4.7). ∎

LEMMA 4.6 [17, p. 248]. *Let* $f \in B_{\sigma,p}$, $1 \le p \le \infty$. *Then*

$$\|\{f(j)\}\|_p \le (1+\pi)\|f\|_p. \tag{4.9}$$

Now we give the proofs of Theorems 1 and 2.

Proof of Theorem 1. Let $1 < p < \infty$, let $f(x) \in L_p(\mathbb{R})$, and let $s_m(x)$, $m = 1, 2, \dots$, $g_\pi(x)$, be the best approximations of $f(x)$ by $\mathscr{S}_{m,p}$ and $B_{\pi,p}$ in $L_p(\mathbb{R})$, respectively. By Lemma 4.6, we have that $\|\{g_\pi(j)\}\|_p < +\infty$; therefore, by virtue of Theorem A, there is a unique $\alpha_m(x) \in \mathscr{S}_{m,p}$ such that $\alpha_m(j) = g_\pi(j)$, $j \in \mathbb{Z}$, and

$$\alpha_m(x) = \sum_{j\in\mathbb{Z}} g_\pi(j) L_m(x-j). \tag{4.10}$$

By the Whittaker−Shannon−Kotelnikov sampling theorem and its generalization [18],

$$g_\pi(x) = \sum_{j\in\mathbb{Z}} g_\pi(j)\frac{\sin\pi(x-j)}{\pi(x-j)}, \qquad \forall x \in \mathbb{R}, \tag{4.11}$$

which together with (4.10) and Theorem B gives

$$\lim_{m\to\infty}\|\alpha_m - g_\pi\|_p = 0, \qquad 1 < p < \infty;$$

therefore,

$$\|f - s_m\|_p \le \|f - \alpha_m\|_p \le \|f - g_\pi\|_p + \|g_\pi - \alpha_m\|_p$$

$$\overline{\lim_{m\to\infty}}\, E(f, \mathscr{S}_{m,p})_p \le E(f, B_{\pi,p})_p + \overline{\lim_{m\to\infty}}\,\|\alpha_m - g_\pi\|_p \le E(f, B_{\pi,p})_p.$$

In the case $p = 1$, and $p = \infty$, in the same way, but using Lemma 4.5, we have

$$\overline{\lim_{m\to\infty}}\, E(f, \mathscr{S}_{m,p})_p \le E(f, B_{\sigma,p})_p, \qquad 0 < \sigma < \pi,\ p = 1, \infty.$$

Now we assume $1 \le p \le \infty$. For fixed $\delta > 0$, let $\beta_{m,\delta}$ be the best approximation of s_m, $m = 1, 2, \ldots,$ by $B_{\pi+\delta,p}$ in $L_p(\mathbb{R})$. Then

$$E(f, B_{\pi+\delta,p})_p \le \|f - \beta_{m,\delta}\|_p \le \|f - s_m\|_p + \|s_m - \beta_{m,\delta}\|_p$$
$$= \|f - s_m\|_p + E(s_m, B_{\pi+\delta,p})_p.$$

Since s_m is a best approximation of $f(x)$ by $\mathscr{S}_{m,p}$ in $L_p(\mathbb{R})$, $\|s_m\|_p \le 2\|f\|_p$, therefore by Theorem 7,

$$\lim_{m\to\infty} E(s_m, B_{\pi+\delta,p})_p = 0;$$

hence, we obtain

$$E(f, B_{\pi+\delta,p})_p \le \varliminf_{m\to\infty} E(f, \mathscr{S}_{m,p})_p. \tag{4.12}$$

For this and Lemma 4.2, we have

$$E(f, B_{\pi,p})_p \le \varliminf_{m\to\infty} E(f, \mathscr{S}_{m,p})_p.$$

Theorem 1 is proved. ▮

Proof of Theorem 2. Since $\mathfrak{M}$ is bounded in $L_p(\mathbb{R})$, $E(\mathfrak{M}, B_{\pi,p})_p < +\infty$. Let $\{\varepsilon_k\}_{k\in\mathbb{N}}$ be a sequence tending to zero and $\{f_k\}_{k\in\mathbb{N}} \subset \mathfrak{M}$, such that

$$E(\mathfrak{M}, B_{\pi+\delta,p})_p \le E(f, B_{\pi+\delta,p})_p + \varepsilon_k. \tag{4.13}$$

Denote by $s_{m,k}$ a best approximation of $f(x)$ by $\mathscr{S}_{m,p}$ in $L_p(\mathbb{R})$, and denote by $\alpha_{m,k}$ a best approximation of $s_{m,k}$ by $B_{\pi+\delta,p}$ in $L_p(\mathbb{R})$; therefore, $\|s_{m,k}\|_p \le 2\|f_k\|_p$. By virtue of Theorem 7, we have

$$\lim_{m\to\infty} \|s_{m,k} - \alpha_{m,k}\|_p = \lim_{m\to\infty} E(s_{m,k}, B_{\pi+\delta,p})_p = 0; \tag{4.14}$$

therefore,

$$\begin{aligned} E(\mathfrak{M}, B_{\pi+\delta,p})_p &\le E(f_k, B_{\pi+\delta,p})_p + \varepsilon_k \le \|f - \alpha_{m,k}\|_p + \varepsilon_k \\ &\le \|f_k - s_{m,k}\|_p + \|s_{m,k} - \alpha_{m,k}\|_p + \varepsilon_k \\ &\le E(\mathfrak{M}, \mathscr{S}_{m,p})_p + E(s_{m,k}, B_{\pi+\delta,p})_p + \varepsilon_k, \end{aligned} \tag{4.15}$$

which together with (4.14) gives

$$E(\mathfrak{M}, B_{\pi+\delta,p})_p \le \varliminf_{m\to\infty} E(\mathfrak{M}, \mathscr{S}_{m,p})_p. \tag{4.16}$$

On the other hand, there is a sequence $\{g_k\} \subset \mathfrak{M}$, such that

$$E(\mathfrak{M}, B_{\pi,p})_p \leq E(g_k, B_{\pi,p})_p + \varepsilon_k, \qquad k = 1, 2, \ldots,$$

from Lemma 4.2,

$$\lim_{\delta \to 0} E(g_k, B_{\pi+\delta,p}) = E(g_k, B_{\pi,p})_p; \tag{4.17}$$

therefore,

$$\begin{aligned} E(\mathfrak{M}, B_{\pi,p})_p &\leq E(g_k, B_{\pi+\delta,p})_p + \varepsilon_k \\ &= \varliminf_{\delta \to 0} E(g_k, B_{\pi+\delta,p})_p + \varepsilon_k \leq \varliminf_{\delta \to 0} E(\mathfrak{M}, B_{\pi+\delta,p})_p + \varepsilon_k \\ &\leq \varlimsup_{\delta \to 0} E(\mathfrak{M}, B_{\pi+\delta,p})_p + \varepsilon_k \leq E(\mathfrak{M}, B_{\pi,p})_p + \varepsilon_k. \end{aligned}$$

Letting $k \to \infty$ in (4.17), we have

$$\lim_{\delta \to 0} E(\mathfrak{M}, B_{\pi+\delta,p})_p = E(\mathfrak{M}, B_{\pi,p})_p. \tag{4.18}$$

Combining (4.16), (4.18), we obtain

$$E(\mathfrak{M}, B_{\pi,p})_p \leq \varliminf_{m \to \infty} E(\mathfrak{M}, \mathscr{S}_{m,p})_p.$$

This completes the proof of Theorem 2. ∎

ACKNOWLEDGMENT

I am grateful to the referees for their valuable comments about this paper.

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